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Upper Bounds for the Zeros of Ultraspherical Polynomials*

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For $k = 1, 2, ..., [n/2]$ let $x_{nk}^{(\lambda)}$ denote the kth positive zero in decreasing order of the ultraspherical polynomial $P_n^{(\lambda)}(x)$. We establish upper bounds for $x_{nk}^{(\lambda)}$. All the bounds become exact when $\lambda = 0$ and, in some cases (see case (iii) of Theorem 3.1), also when $\lambda = 1$. As a consequence of our results, we obtain for the largest zero $x_{n1}^{(\lambda)} < (n^2 + 2\lambda n)^{1/2}/(n + \lambda)$, $\lambda > 0$. We point out that our results remain useful for large values of λ. Numerical examples show that our upper bounds are quite sharp.

1. INTRODUCTION

For $\lambda > -\frac{1}{2}$ we denote by $x_{nk}^{(\lambda)}$ the kth zero, in decreasing order, of the ultraspherical polynomial $P_n^{(\lambda)}(x)$.

In the literature there are many inequalities for $x_{nk}^{(\lambda)}$. The most stringent results are obtained involving the positive zeros $j_{\nu k}$ $(k = 1, 2, ...)$ of the Bessel function of the first kind. We recall, for example, the well known bounds [3, p. 127]

$$\frac{j_{\nu k}}{[(n+\lambda)^2 + K\lambda(1-\lambda)]^{1/2}} < \theta_{nk}^{(\lambda)} < \frac{j_{\nu k}}{n+\lambda}, \qquad k = 1, 2, ..., n,$$

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where $\theta_{nk}^{(\lambda)} = \arccos x_{nk}^{(\lambda)}$, $\nu = \lambda - \frac{1}{2}$, and K is a positive numerical constant. These inequalities are valid only for $0 < \lambda < 1$; when $\lambda > 1$ they must be reversed [1, p. 216].

In this paper we are concerned with some bounds for $x_{nk}^{(\lambda)}$. Due to the symmetry relation $P_n^{(\lambda)}(-x) = (-1)^n P_n^{(\lambda)}(x)$ we can confine ourself to the positive zeros $x_{nk}^{(\lambda)}$ with $k = 1, 2, ..., [n/2]$.

The starting point of our investigations is the differential equation [3, p. 81].

$$y'' + \left[\frac{(n+\lambda)^2}{1-x^2} + \frac{1/2 + \lambda - \lambda^2 + x^2/4}{(1-x^2)^2}\right] y = 0 \tag{1.1}$$

satisfied by $u_\lambda(x) = (1-x^2)^{\lambda/2 + 1/4} P_n^{(\lambda)}(x)$; the main tool used is the Sturm comparison theorem [3, p, 19].

2. Preliminaries

In this section we study some properties of the function

$$\varphi(x) = \int_0^x \frac{(A - Bs^2)^{1/2}}{1-s^2}\, ds, \qquad 0 < A < B,\ |x| \leqq \sqrt{A/B}, \tag{2.1}$$

which will play an important role in our investigations. Clearly $\varphi(x)$ is a strictly increasing function. The values of A and B will be specified later on. After the substitutions

$$x = \sqrt{A/B} \sin \theta, \qquad s = \sqrt{A/B} \sin t, \qquad |\vartheta| < \pi/2 \tag{2.2}$$

the integral (2.1) can be written in the form

$$\begin{aligned} \varphi(x) &= \frac{A}{\sqrt{B}} \int_0^\theta \frac{\cos^2 t}{1 - (A/B)\sin^2 t}\, dt \\ &= \sqrt{B}\left[\theta - \sqrt{\frac{B-A}{B}} \arctan\left(\sqrt{\frac{B-A}{B}} \tan \theta\right)\right] = \sqrt{B}\, I(\theta). \end{aligned} \tag{2.3}$$

Lemma 2.1. *Let $\rho(x)$ be defined by*

$$\rho(x) = [\varphi'(x)]^{-1/2}. \tag{2.4}$$

Then the functions

$$v_1(x) = \rho(x) \cos \varphi(x), \qquad v_2(x) = \rho(x) \sin \varphi(x) \tag{2.5}$$

are solutions of the differential equation

$$v''+\left(\varphi'^2-\frac{\rho''}{\rho}\right)v=0, \qquad |x|<\sqrt{A/B}. \tag{2.6}$$

Proof. Lemma 2.1 can be easily verified; we omit the details of the proof.

By (2.5) the function $v_1(x)$ has zeros where $\varphi(x)=\pm\pi/2, \pm 3\pi/2, \ldots$. Suppose first n even and denote by $\hat{x}_{nk}^{(\lambda)}$ the solutions of the equation

$$\varphi(\hat{x}_{nk}^{(\lambda)})=(n+1-2k)\,\pi/2, \qquad k=1,2,\ldots,n, \tag{2.7}$$

provided that they exist. To ensure the existence of all $\hat{x}_{nk}^{(\lambda)}$ we need the relation

$$\varphi(\sqrt{A/B})\geqq(n-1)\,\pi/2. \tag{2.8}$$

Clearly the values $\hat{x}_{nk}^{(\lambda)}$ $(k=1,2,\ldots,n)$ are zeros of $v_1(x)$ when n is even. Similarly, it is easy to check that $\hat{x}_{nk}^{(\lambda)}$ defined by (2.7) are zeros of $v_2(x)$ when n is odd. Using (2.3) and (2.2) relation (2.7) can be written as

$$\hat{x}_{nk}^{(\lambda)}=\sqrt{A/B}\sin\theta_{nk}^{(\lambda)}=\sqrt{A/B}\sin I^{-1}\left(\frac{n+1-2k}{\sqrt{B}}\frac{\pi}{2}\right), \qquad k=1,2,\ldots,n. \tag{2.9}$$

Now we want to choose the values A and B in such a way that the differential equation (1.1) is a Sturmian majorant for (2.6) on the interval $(-\sqrt{A/B}, \sqrt{A/B})$. In order to have this we set

$$\begin{aligned} A&=n^2+2\lambda n+\varepsilon \\ B&=(n+\lambda)^2+\delta \end{aligned} \tag{2.10}$$

and consider the difference between the coefficients of the differential equations (1.1) and (2.6)

$$\frac{(n+\lambda)^2}{1-x^2}+\frac{1/2+\lambda-\lambda^2+x^2/4}{(1-x^2)^2}-\left(\varphi'^2-\frac{\rho''}{\rho}\right), \qquad |x|<\sqrt{A/B}. \tag{2.11}$$

From (2.1) we get

$$\frac{\rho'}{\rho}=-\frac{1}{2}\frac{\varphi''}{\varphi'}=-\frac{1}{2}(\log\varphi')'=\frac{Bx}{2(A-Bx^2)}-\frac{x}{1-x^2} \tag{2.12}$$

and differentiating we obtain

$$\left(\frac{\rho'}{\rho}\right)' = \frac{\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2 = -\frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)'$$

and

$$\frac{\rho''}{\rho} = \frac{3B^2x^2 + 2AB}{4(A - Bx^2)^2} - \frac{Bx^2}{(1 - x^2)(A - Bx^2)} - \frac{1}{(1 - x^2)^2}.$$

Therefore (2.11) can be written as

$$\frac{G(x)}{4(1 - x^2)^2\,(A - Bx^2)^2},$$

wnere

$$\begin{aligned} G(x) = {} & 4B^2\delta x^6 + [4B^2(\lambda - \varepsilon) - 4B^2 - 8AB\delta + 4AB]\, x^4 \\ & + [4A^2\delta - 8AB(\lambda - \varepsilon) + A^2 - 4AB + 3B^2]\, x^2 \\ & + 4A^2(\lambda - \varepsilon) + 2AB - 2A^2. \end{aligned} \tag{2.13}$$

The differential equation (1.1) is a Sturmian majorant of (2.6) if $G(x) > 0$ for $|x| < \sqrt{A/B}$.

In the following Lemma we give three examples of choice for ε and δ such that $G(x) > 0$.

LEMMA 2.2. *The function $G(x)$ defined by* (2.13) *is positive on the interval $(-\sqrt{A/B}, \sqrt{A/B})$ in the following cases*

(i) $\varepsilon = \delta = 0$, $\lambda > 0$

(ii) $\varepsilon = \delta = \lambda$, $\lambda > 0$

(iii) $\varepsilon = \lambda$, $\delta = 0$, $-\frac{1}{2} < \lambda < 0$ *or* $\lambda \geqq 1$.

Proof. Case (i): Since $B - A = \lambda^2$ from (2.13) we have

$$\begin{aligned} G(x) &= 4(B^2\lambda - B^2 + AB)\, x^4 + (-8AB\lambda + A^2 - 4AB + 3B^2)\, x^2 \\ &\quad + 4A^2\lambda + 2AB - 2A^2 \\ &= 4\lambda(A - Bx^2)^2 + \lambda^2 q(x^2), \end{aligned} \tag{2.14}$$

where

$$q(t) = -4Bt^2 + (3B - A)\, t + 2A. \tag{2.15}$$

The first term on the right-hand side of (2.14) is clearly positive; thus to prove that $G(x) > 0$ it is sufficient to show that $q(t) > 0$ when $0 \leqq t < A/B$.

It is clear that $q(0) > 0$ and $q(A/B) = 5A(B-A)/B > 0$. Moreover $q(t)$ is concave and we conclude that $q(t) > 0$ for all $t \in [0, A/B]$.

Case (ii): As in case (i) we have $B - A = \lambda^2$ and

$$G(x) = 4\lambda x^2(A - Bx^2)^2 + \lambda^2 q(x^2),$$

where $q(t)$ is defined by (2.15). As before we conclude that $G(x) > 0$.

Case (iii): Now we have $B - A = \lambda^2 - \lambda$ and

$$G(x) = (\lambda^2 - \lambda)\, q(x^2)$$

which is positive for $-\frac{1}{2} < \lambda \leqq 0$ and $\lambda > 1$.

The proof of Lemma 2.2 is complete.

3. The Main Result

Now we are in the position to prove the main result of the paper.

THEOREM 3.1. *For $k = 1, 2, ..., [n/2]$ let $x_{nk}^{(\lambda)}$ be the kth positive zero in decreasing order of the ultraspherical polynomial $P_n^{(\lambda)}(x)$. Then the inequalities*

$$x_{nk}^{(\lambda)} \leqq \hat{x}_{nk}^{(\lambda)}, \qquad n = 1, 2, ..., \qquad k = 1, 2, ..., [n/2] \tag{3.1}$$

hold for the following choices of A and B in (2.10):

(i) $A = n^2 + 2\lambda n$, $B = (n+\lambda)^2$, $\lambda > 0$

(ii) $A = n^2 + 2\lambda n + \lambda$, $B = (n+\lambda)^2 + \lambda$, $\lambda > 0$

(iii) $A = n^2 + 2\lambda n + \lambda$, $B = (n+\lambda)^2$, $-\frac{1}{3} \leqq \lambda < 0$ *or* $\lambda \geqq 1$.

When $\lambda = 0$ the ultraspherical polynomial reduce to the Tchebycheff polynomial and we have equality in (3.1) in all cases. If $\lambda = 1$ we have equality in (iii).

Proof. To ensure the existence of $\hat{x}_{nk}^{(\lambda)}$ we need to prove inequality (2.8). By (2.3) we have $\varphi(\sqrt{A/B}) = (\pi/2)(\sqrt{B} - \sqrt{B-A})$. Actually we have in case (1)

$$\varphi(\sqrt{A/B}) = n\,\frac{\pi}{2} > (n-1)\,\frac{\pi}{2}$$

and in the case (ii)

$$\varphi(\sqrt{A/B} = [\sqrt{(n+\lambda)^2 + \lambda} - \lambda]\,\frac{\pi}{2} \geqq n\,\frac{\pi}{2} > (n-1)\,\frac{\pi}{2}.$$

Finally, in case (iii), we get for $\lambda \geqq -1/3$

$$\varphi(\sqrt{A/B}) = (n + \lambda - \sqrt{\lambda^2 - \lambda})\frac{\pi}{2} \geqq (n-1)\frac{\pi}{2}.$$

We distinguish the cases of even n and odd n. We consider only the first case in detail. The second one can be discussed in a similar way.

For even values of n we have

$$P_n^{(\lambda)}(0) \neq 0, \qquad \frac{d}{dx} P_n^{(\lambda)}(x)\bigg|_{x=0} = 0.$$

Let us consider the solution $v_1(x) = \rho \cos \varphi$ of Eq. (2.6). By (2.1), (2.4), and (2.12) we have

$$\varphi(0) = 0, \qquad \rho(0) = \sqrt{A} > 0, \qquad \rho'(0) = 0,$$

therefore $v_1(0) > 0$ and $v_1'(0) = 0$. Moreover by Lemma 2.2 the differential equation (1.1) is a Sturmian majorant of (2.6). Hence an application of the Sturm comparison theorem [3, p. 19] at the point $x = 0$ gives that the kth positive zero of $P_n^{(\lambda)}(x)$ occurs before the kth positive zero of $v_1(x)$, which proves the first part of the theorem. A similar argument applies when n is odd.

The formula (2.9) for $\hat{x}_{nk}^{(\lambda)}$ enables us to obtain rather good upper bounds for $x_{nk}^{(\lambda)}$, but unfortunately its application is not immediate. Therefore it is useful to report here some useful consequences of Theorem 3.1 that we establish as corollaries.

COROLLARY 3.1. *Let A and B satisfy the conditions of Theorem 3.1. Then the inequalities*

$$x_{nk}^{(\lambda)} \leqq \sqrt{A/B} \sin\left(\frac{n+1-2k}{\sqrt{B} - \sqrt{A/B}} \frac{\pi}{2}\right), \qquad k = 1, 2, ..., [n/2] \tag{3.2}$$

hold.

Proof. Since $\sqrt{(B-A)/B} < 1$ by (2.3) we have

$$\varphi(x) \geqq \sqrt{B}(\theta - \sqrt{(B-A)/B}\,\theta)$$

or

$$\theta < \frac{\varphi(x)}{\sqrt{B} - \sqrt{B-A}}.$$

Consequently by (2.2)

$$x \leqq \sqrt{A/B} \sin \frac{\varphi(x)}{\sqrt{B} - \sqrt{B-A}}. \tag{3.3}$$

Moreover, by Theorem 3.1, we have $\varphi(x_{nk}^{(\lambda)}) \leqq \varphi(\hat{x}_{nk}^{(\lambda)})$ (because $\varphi(x)$ is a strictly increasing function). Applying (3.3) with $x = x_{nk}^{(\lambda)}$ and taking into account (2.7) we obtain the desired result.

Remark. By (3.2) and Theorem 3.1 we obtain immediately the interesting bound for the largest zero

$$x_{nl}^{(\lambda)} < \frac{\sqrt{n^2 + 2\lambda n}}{n + \lambda}, \qquad \lambda \geqq 0.$$

Theorem 3.1 gives stringent upper bounds for $x_{nk}^{(\lambda)}$. This and other things will be shown in the next section.

Now we use the result (2.3) to derive the first three terms of the series expansion of $I^{-1}(\theta)$. In this way we get approximations for $x_{nk}^{(\lambda)}$ which will be precise particularly when k is near to $n/2$. For the sake of simplicity we use the notation

$$m = \sqrt{(B-A)/B}.$$

Then $0 < m < 1$ and by (2.3)

$$I(\theta) = \theta - m \arctan(m \tan \theta).$$

From the expansions of $\tan \theta$ and $\arctan \theta$ we find

$$I(\theta) = (1 - m^2)\, \theta - \frac{m^2}{3} (1 - m^2)\, \theta^3 - \frac{m^2(1 - m^2)(2 - 3m^2)}{15} \theta^5 + \cdots$$

and by straightforward calculations we obtain

$$I^{-1}(\theta) = \frac{\theta}{1 - m^2} + \frac{m^2}{3} \left(\frac{\theta}{1 - m^2}\right)^3 + \frac{2m^2(1 + m^2)}{15} \left(\frac{\theta}{1 - m^2}\right)^5 + \cdots. \tag{3.4}$$

Now we enunciate the result, which gives approximations of $x_{nk}^{(\lambda)}$.

COROLLARY 3.2. *Let A, B and λ satisfy the conditions of Theorem 3.1. Then for $x_{nk}^{(\lambda)}$ we have the approximation formula*

$$x_{nk}^{(\lambda)} \doteq \sqrt{A/B} \left[\tau + \frac{2m^2 - 1}{6} \tau^3 + \frac{16m^4 - 4m^2 + 1}{120} \tau^5 \right], \tag{3.5}$$

where

$$\tau = (n+1-2k)\frac{\sqrt{B}}{A}\frac{\pi}{2}, \qquad m^2 = \frac{B-A}{B}.$$

Proof. By (2.9) we have to compute the first three terms of the series expansion of $\sin I^{-1}(\theta)$ with $\theta = (n+1-2k)\,\pi/\sqrt{B}\,2$. Using Formula (3.4) for $I^{-1}(\theta)$ we find the desired expansion.

4. Numerical Applications

In this section we make some applications of the results established in the previous section.

According to case (iii) of Theorem 3.1 we choose

$$A = n^2 + 2\lambda n + \lambda, \qquad B = (n+\lambda)^2.$$

When $\lambda = 1.1$ and $n = 15$ we have

$$A = 259.1, \qquad B = 259.21.$$

With these values of A and B we compute upper bounds $\hat{x}_{nk}^{(\lambda)}$ by Formula (2.9), applying the Newton–Raphson method. We have to compute the function $I^{-1}(\theta)$.

Let θ be defined by

$$\theta = I^{-1}(t).$$

This equation is equivalent to

$$t = I(\theta),$$

TABLE I

$\lambda = 1.1$, $n = 15$, Case (iii)

k	$x_{15,k}^{(1.1)}$	$\hat{x}_{15,k}^{(1.1)}$
1	0.97928446	0.97929259
2	0.92124789	0.92125024
3	0.82824118	0.82824222
4	0.70379775	0.70379831
5	0.55264140	0.55264172
6	0.38050941	0.38050959
7	0.19393512	0.19393520

TABLE II

$\lambda = 4,\ n = 15$

k	Theorem 3.1 case (i)	Theorem 3.1 case (ii)	Theorem 3.1 case (iii)	Exact values	Corollary 3.1 case (iii)	Corollary 3.2 case (iii)
1	0.9394	0.93629	0.932630	0.932375	0.9714	0.932472
2	0.8572	0.85387	0.851875	0.851787	0.9211	0.851993
3	0.7530	0.74957	0.748449	0.748405	0.8322	0.748474
4	0.6284	0.62533	0.624760	0.624734	0.7114	0.624766
5	0.4870	0.48433	0.484090	0.484073	0.5606	0.484090
6	0.3322	0.33031	0.330242	0.330233	0.3869	0.330242
7	0.1684	0.16741	0.167401	0.167397	0.1975	0.167401

where $I(\theta)$ is given by (2.3). We compute the value of θ as a solution of this equation. We observe that the function $I(\theta)$ is strictly increasing and concave. In fact

$$\frac{d}{d\theta} I(\theta) = 1 - \frac{m^2}{\cos^2 \theta + m^2 \sin^2 \theta} = \frac{(1 - m^2) \cos^2 \theta}{\cos^2 \theta + m^2 \sin^2 \theta} > 0.$$

The approximate values of $\hat{x}_{nk}^{(\lambda)}$ are displayed in Table I together with the exact values $x_{nk}^{(\lambda)}$ taken by [2].

As a next example we consider the case

$$n = 15, \qquad \lambda = 4$$

and take into consideration all the three cases of Theorem 3.1, the result of Corollary 3.1, and the expansion established in Corollary 3.2. The results are compared with the exact value of $x_{15,k}^{(4)}$, $k = 1, 2, ..., 7$.

Table II shows that Corollary 3.2 gives good starting values to compute $x_{nk}^{(\lambda)}$. We observe that the most precise upper bounds are obtained in case (iii) of Theorem 3.1.

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